Dipper-James-Murphy's conjecture for Hecke algebras of type B

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DIPPER-JAMES-MURPHY'S CONJECTURE FOR HECKE ALGEBRAS OF TYPE $\mathbf{B_n}$

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Abstract. We prove a conjecture by Dipper, James and Murphy that a bipartition is restricted if and only if it is Kleshchev. Hence the restricted bipartitions naturally label the crystal graphs of level two irreducible integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules and the simple modules of Hecke algebras of type B_n.

Dedicated to Toshiaki Shoji and Ken-ichi Shinoda on their 60th birthdays

1. Introduction

Let F be a field, q and Q invertible elements of F. The Hecke algebra of type B_n is the F-algebra defined by generators $T_0, \dots, T_{n-1}$ and relations

$$(T_0 - Q)(T_0 + 1) = 0, \quad (T_i - q)(T_i + 1) = 0 \ (1 \leqslant i < n)$$
$$(T_0T_1)^2 = (T_1T_0)^2, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \ (1 \leqslant i < n-1)$$
$$T_iT_j = T_jT_i \ (j \geq i+2).$$

We denote it by $\mathcal{H}_n(Q,q)$, or $\mathcal{H}_n$ for short. The representation theory of $\mathcal{H}_n$ in the semisimple case was studied by Hoefsmit, which had applications in determining generic degrees and Lusztig's a-values. Motivated by the modular representation theory of $U_n(q)$ and $Sp_{2n}(q)$ in the non-defining characteristic case, Dipper, James and Murphy began the study of the modular case more than a decade ago. The first task was to obtain classification of simple modules. For this, they constructed Specht modules which are indexed by the set of bipartitions [7]. The work shows in particular that Hecke algebras of type B_n are cellular algebras in the sense of Graham and Lehrer. [1] Then they conjectured that the simple modules were labeled by (Q,e)-restricted bipartitions. Their philosophy to classify the simple $\mathcal{H}_n$-modules resembles the highest weight theory in Lie theory: let $\mathfrak{g}$ be a semisimple Lie algebra. It has a commutative Lie subalgebra $\mathfrak{h}$, the Cartan subalgebra. One dimensional $\mathfrak{h}$-modules are called weights (by abuse of notion). When a $\mathfrak{g}$-module admits a simultaneous generalized eigenspace decomposition with respect to $\mathfrak{h}$, the decomposition is called the (generalized) weight space decomposition. Let Λ be a weight. Suppose that a $\mathfrak{g}$-module M has the property that

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[1]This result has been recently generalized by Geck in [8].

(i) Λ appears in the weight space decomposition of M,
(ii) If N is a proper $\mathfrak{g}$-submodule of M then Λ does not appear in the weight space decomposition of N.

Then the standard argument shows that M has a unique nonzero irreducible quotient. In fact, Verma modules enjoy the property and their irreducible quotients give a complete set of simple objects in the BGG category. Now we turn to the Hecke algebra $\mathcal{H}_n$. Define the Jucys-Murphy elements $t_1, \dots, t_n$ by $t_1 = T_0$ and $t_{i+1} = q^{-1}T_i t_i T_i$, for $1 \leqslant i \leqslant n-1$. They generate a commutative subalgebra $\mathcal{A}_n$ of the Hecke algebra $\mathcal{H}_n$, and $\mathcal{A}_n$ plays the role of the Cartan subalgebra: one dimensional $\mathcal{A}_n$-modules are called *weights* and the generalized simultaneous eigenspace decomposition of an $\mathcal{H}_n$-module is called the *weight space decomposition*. Any weight is uniquely determined by the values at $t_1, \dots, t_n$ of the weight, and the sequence of these values in this order is called the *residue sequence*. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition (see §2.1) and let $\mathbf{t}$ be a standard bitableau of shape $\boldsymbol{\lambda}$ (see Def. 2.9). Then, $\mathbf{t}$ defines a weight whose values at t_i are given by $c_i q^{b_i - a_i}$ where a_i and b_i are the row number and the column number of the node of $\mathbf{t}$ labelled by i respectively, $c_i = -Q$ if the node is in $\lambda^{(1)}$ and $c_i = 1$ if the node is in $\lambda^{(2)}$. By the theory of seminormal representations in the semisimple case and the modular reduction, a weight appears in some $\mathcal{H}_n$-module if and only if it is obtained from a bitableau this way.

Suppose that there is a weight obtained from a bitableau $\mathbf{t}$ of shape $\boldsymbol{\lambda}$ such that it does not appear in $S^{\boldsymbol{\mu}}$ when $\boldsymbol{\mu} \lhd \boldsymbol{\lambda}$. If such a bitableau exists, we say that $\boldsymbol{\lambda}$ is (Q, e)-*restricted*. This is a clever generalization of the notion of e-restrictedness. Recall that a partition $\lambda = (\lambda_0, \lambda_1, \dots)$ is called e-*restricted* if $\lambda_{i+1} - \lambda_i < e$, for all $i \geq 0$. Recall also that we have the similar Specht module theory for Hecke algebras of type A. Using Jucys-Murphy elements of the Hecke algebra of type A, we can define weights as well. Then, a partition is e-restricted if and only if there is a weight obtained from a tableau of shape λ such that it does not appear in S^{μ} when $\mu \lhd \lambda$.

Recall from [7] that

$$[S^{\boldsymbol{\lambda}}] = [D^{\boldsymbol{\lambda}}] + \sum_{\boldsymbol{\mu} \lhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}} [D^{\boldsymbol{\mu}}],$$

where the summation is over $\boldsymbol{\mu}$ such that $D^{\boldsymbol{\mu}} \neq 0$, $d_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ are decomposition numbers, and $\sum_{\boldsymbol{\mu} \lhd \boldsymbol{\lambda}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}} [D^{\boldsymbol{\mu}}]$ is represented by the radical of the bilinear form on $S^{\boldsymbol{\lambda}}$. As $D^{\boldsymbol{\mu}}$ is a surjective image of $S^{\boldsymbol{\mu}}$, it implies that the weight does not appear in the radical, while it appears in $S^{\boldsymbol{\lambda}}$. Therefore, $D^{\boldsymbol{\lambda}} \neq 0$ if $\boldsymbol{\lambda}$ is (Q, e)-restricted. Unlike the case of the BGG category, we may have $D^{\boldsymbol{\lambda}} = 0$ and it is important to know when it occurs. When $-Q$ is not a power of q, a bipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is (Q, e)-restricted if and only if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. Thus we know when a bipartition is (Q, e)-restricted. Further, [6, Thm 4.18] implies that $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is (Q, e)-restricted, that is, simple $\mathcal{H}_n$-modules are labelled by (Q, e)-restricted bipartitions. Now we suppose that $-Q$ is a power of q. More precisely, we suppose that

(a) q is a primitive e^{th} root of unity with $e \geqslant 2$,
(b) $-Q = q^m$, for some $0 \leqslant m < e$.

in the rest of the paper. We call (Q, e)-restricted bipartitions *restricted* bipartitions. They conjectured in this case that $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is restricted, and it has been known as the Dipper-James-Murphy conjecture for Hecke algebras of type B_n.

Later, connection with the theory of canonical bases in deformed Fock spaces in the sense of Hayashi and Misra-Miwa was discovered by Lascoux-Leclerc-Thibon [11] and its proof in the framework of cyclotomic Hecke algebras [1] allowed the first author and Mathas [3] [5] to label simple $\mathcal{H}_n$-modules by the n^{th} layer of the crystal graph of the level two irreducible integrable $\mathfrak{g}(A^{(1)}_{e-1})$-module $L_v(\Lambda_0 + \Lambda_m)$. In the theory, the crystal graph is realized as a subcrystal of the crystal of bipartitions, and the nodes of the crystal graph are called *Kleshchev* bipartitions. More precise definition is given in the next section and $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is Kleshchev if and only if $\lambda^{(2)} \otimes \lambda^{(1)}$ belongs to the subcrystal $B(\Lambda_0 + \Lambda_m)$ of $B(\Lambda_0) \otimes B(\Lambda_m)$, where the crystals $B(\Lambda_0)$ and $B(\Lambda_m)$ are realized on the set of e-restricted partitions. Now, $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is Kleshchev by [3]. Hence, we obtained the classification of simple $\mathcal{H}_n$-modules, or more precisely description of the set $\{\boldsymbol{\lambda} \mid D^{\boldsymbol{\lambda}} \neq 0\}$, through a different approach and the Dipper-James-Murphy conjecture in the modern language is the statement that the Kleshchev bipartitions are precisely the restricted bipartitions.

The aim of this paper is to prove the Dipper-James-Murphy conjecture. Recall that Lascoux, Leclerc and Thibon considered Hecke algebras of type A and they showed that if λ is a e-restricted partition then we can find $a_1, \dots, a_p$ and $i_1, \dots, i_p$ such that we may write

$$f_{i_1}^{(a_1)} \dots f_{i_p}^{(a_p)} \emptyset = \lambda + \sum_{\nu \rhd \lambda} c_{\nu,\lambda}(v) \nu$$

in the deformed Fock space, where $c_{\nu,\lambda}(v)$ are Laurent polynomials. This follows from the ladder decomposition of a partition. Then LLT algorithm proves that Kleshchev partitions are precisely e-restricted partitions. The second author [10] proved the similar formula for FLOTW multipartitions in the Jimbo-Misra-Miwa-Okado higher level Fock space using certain a-values instead of the dominance order. Recall that Geck and Rouquier gave another method to label simple $\mathcal{H}_n$-modules by bipartitions. The result shows that the parametrizing set of simple $\mathcal{H}_n$-modules in the Geck-Rouquier theory, which is called the *canonical basic set*, is precisely the set of the FLOTW bipartitions. Our strategy to prove the conjecture is to give the analogous formula for Kleshchev bipartitions. To establish the formula, a non-recursive characterization of Kleshchev bipartitions given by the first author, Kreiman and Tsuchioka [4] plays a key role.

The paper is organized as follows. In the first section, we briefly recall the definition of Kleshchev bipartitions. We also recall the main result of [4]. In the second section, we use this result to give an analogue for bipartitions of the ladder decomposition. Finally, the last section gives a proof for the conjecture.

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2. Preliminaries

In this section, we recall the definition of Kleshchev bipartitions together with the main result of [4] which gives a non-recursive characterization of these bipartitions. We fix m as in the inroduction. Namely, the parameter Q of the Hecke algebra is $Q = -q^m$ with $0 \leqslant m < e$.

2.1. **First definitions.** Recall that a partition λ is a sequence of weakly decreasing nonnegative integers $(\lambda_0, \lambda_1, \cdots)$ such that $|\lambda| = \sum_{i \geq 0} \lambda_i$ is finite. If $\lambda_i = 0$ for $i \geq r$ then we write $\lambda = (\lambda_0, \cdots, \lambda_{r-1})$. A *bipartition* $\boldsymbol{\lambda}$ is an ordered pair of partitions $(\lambda^{(1)}, \lambda^{(2)})$. $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + |\lambda^{(2)}|$ is called the *rank* of $\boldsymbol{\lambda}$. The empty bipartition $(\emptyset, \emptyset)$ is the only bipartition of rank zero. The *diagram* of $\boldsymbol{\lambda}$ is the set

$$\{(a, b, c) \mid 1 \leqslant c \leqslant 2, 0 \leqslant b \leqslant \lambda_a^{(c)} - 1\} \subseteq \mathbb{Z}^3_{\geq 0}.$$

We often identify a bipartition with its diagram. The *nodes* of $\boldsymbol{\lambda}$ are the elements of the diagram. Let $\gamma = (a, b, c)$ be a node of $\boldsymbol{\lambda}$. Then the *residue* of γ is defined by

$$\operatorname{res}(\gamma) = \begin{cases} b - a + m \pmod{e} & \text{if } c = 1, \\ b - a \pmod{e} & \text{if } c = 2. \end{cases}$$

By assigning residues to the nodes of a bipartition, we view a bipartition as a colored diagram with colors in $\mathbb{Z}/e\mathbb{Z}$.

Example 2.1. Put $e = 4$, $m = 2$ and $\boldsymbol{\lambda} = ((3, 2), (4, 2, 1))$. Then the colored diagram associated with $\boldsymbol{\lambda}$ is as follows.

$$\left(\begin{array}{|c|c|c|} \hline 2 & 3 & 0 \\ \hline 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \;,\; \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 3 & 0 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \right)$$

If γ is a node with residue i, we say that γ is an *i-node*. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two bipartitions such that $\boldsymbol{\mu} = \boldsymbol{\lambda} \sqcup \{\gamma\}$. Then, we denote $\boldsymbol{\mu}/\boldsymbol{\lambda} = \gamma$ and if $\operatorname{res}(\gamma) = i$, we say that γ is a *removable i-node* of $\boldsymbol{\mu}$. We also say that γ is an *addable i-node* of $\boldsymbol{\lambda}$ by abuse of notion. [2]

Let $i \in \mathbb{Z}/e\mathbb{Z}$. We choose a total order on the set of removable and addable i-nodes of a bipartition. Let $\gamma = (a, b, c)$ and $\gamma' = (a', b', c')$ be removable

[2] An addable i-node of $\boldsymbol{\lambda}$ is not a node of $\boldsymbol{\lambda}$.

or addable i-nodes of a bipartition. We say that γ is *above* γ' if either $c = 1$ and $c' = 2$, or $c = c'$ and $a < a'$. [3]

Let $\mathcal{F}$ be the vector space over $\mathbb{Q}$ such that the basis is given by the set of all bipartitions. We color the nodes of bipartitions as above. We call it the (level two) *Fock space*. We may equip it with $\hat{sl}_e$-module structure in which the action of the Chevalley generators is given by

$$e_i\boldsymbol{\lambda} = \sum_{\boldsymbol{\nu}:\mathrm{res}(\boldsymbol{\lambda}/\boldsymbol{\nu})=i} \boldsymbol{\nu}, \quad f_i\boldsymbol{\lambda} = \sum_{\boldsymbol{\nu}:\mathrm{res}(\boldsymbol{\nu}/\boldsymbol{\lambda})=i} \boldsymbol{\nu}.$$

Using the total order on the set of removable and addable i-nodes given above, we deform the $\hat{sl}_e$-module structure to $U_v(\hat{sl}_e)$-module structure on the deformed Fock space $\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}(v)$, which is the tensor product of two level one deformed Fock spaces. We refer to [2] for the details.

2.2. **Kleshchev bipartitions.** Recall that the crystal basis of the deformed Fock space is given by the basis vectors of the deformed Fock space. Hence it defines a crystal structure on the set of bipartitions. We call it *the crystal of bipartitions*. As is explained in [2], the map $(\lambda^{(1)}, \lambda^{(2)}) \mapsto \lambda^{(2)} \otimes \lambda^{(1)}$ identifies the crystal of bipartitions with the tensor product of the crystal of partitions of highest weight Λ_0 and that of highest weight Λ_m. As is already mentioned in the introduction, Kleshchev bipartitions are those bipartitions which belongs to the same connected component as the empty bipartition in the crystal of bipartitions. Equivalently, Kleshchev bipartitions are those bipartitions which may be obtained from the empty bipartition by applying the Kashiwara operators successively. Rephrasing it in combinatorial terms, we have a recursive definition of Kleshchev bipartitions as follows.

Let $\boldsymbol{\lambda}$ be a bipartition and let γ be an i-node of $\boldsymbol{\lambda}$, we say that γ is a *normal* i-node of $\boldsymbol{\lambda}$ if, whenever η is an addable i-node of $\boldsymbol{\lambda}$ below γ, there are more removable i-nodes between η and γ than addable i-nodes between η and γ. If γ is the highest normal i-node of $\boldsymbol{\lambda}$, we say that γ is a *good* i-node of $\boldsymbol{\lambda}$. When γ is a good i-node, we denote $\boldsymbol{\lambda} \setminus \{\gamma\}$ by $\tilde{e}_i\boldsymbol{\lambda}$.

Definition 2.2. A bipartition $\boldsymbol{\lambda}$ is *Kleshchev* if either $\boldsymbol{\lambda} = (\emptyset, \emptyset)$ or there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a good i-node γ of $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} \setminus \{\gamma\}$ is Kleshchev.

Note that the definition depends on m. The reader can prove easily using induction on $n = |\lambda^{(1)}| + |\lambda^{(2)}|$ that if $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is Kleshchev then both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. By general property of crystal bases, the following is clear.

Lemma 2.3. *Suppose that $\boldsymbol{\lambda}$ is a Kleshchev bipartition, γ a good i-node of $\boldsymbol{\lambda}$, for some i. Then $\tilde{e}_i\boldsymbol{\lambda} = \boldsymbol{\lambda} \setminus \{\gamma\}$ is Kleshchev.*

[3] We now know that there are more than one Specht module theory, and different Specht module theories prefer different total orders on the set of i-nodes of a bipartition. Our choice of the total order is the one prefered by Dipper-James-Murphy's Specht module theory.

In [4], the first author, Kreiman and Tsuchioka have given a different characterization of Kleshchev bipartitions.

Let λ be a partition. Then the set of *beta numbers of charge* h, where we only use $h = 0$ or $h = m$ in the paper, is by definition the set J_h of decreasing integers

$$j_0 > j_1 > \cdots > j_k > \cdots$$

defined by $j_k = \lambda_k + h - k$, for $k \geqslant 0$. The charge h also defines a coloring of nodes: $\mathrm{res}(\gamma) = b - a + h \pmod{e}$ where a and b are the row number and the column number of a node γ, respectively.

An addable i-node of λ corresponds to $x \in J_h$ such that $x + e\mathbb{Z} = i$ and $x+1 \notin J_h$. We call x an addable i-node of J_h. Similarly, a removable i-node of λ corresponds to $x \in J_h$ such that $x + e\mathbb{Z} = i+1$ and $x - 1 \notin J_h$. We call x a removable i-node of J_h.

We define the abacus display of J_h in the usual way. The i^{th} runner of the abacus is $\{x \in \mathbb{Z} \mid x + e\mathbb{Z} = i\}$, for $i \in \mathbb{Z}/e\mathbb{Z}$.

Definition 2.4. Let λ be an e-restricted partition, $J_m(\lambda)$ the corresponding set of beta numbers of charge m. We write J_m for $J_m(\lambda)$ and define

$$U(J_m) = \{x \in J_m \mid x - e \notin J_m\}.$$

If λ is an e-core then we define $\mathrm{up}_m(\lambda) = \lambda$. Otherwise let $p = \max U(J_m)$ and define

$$V(J_m) = \{x > p \mid x \neq p \pmod{e}, x - e \in J_m, x \notin J_m\}.$$

Note that $V(J_m)$ is nonempty since λ is e-restricted. Let $q = \min V(J_m)$. Then we define

$$\mathrm{up}(J_m) = (J_m \setminus \{p\}) \sqcup \{q\}$$

and we denote the corresponding partition by $\mathrm{up}_m(\lambda)$.

In [4], it is shown that $\mathrm{up}_m(\lambda)$ is again e-restricted and we reach an e-core after applying up_m finitely many times.

Definition 2.5. Let λ be an e-restricted partition. Apply up_m repeatedly until we reach an e-core. We denote the resulting e-core by $\mathrm{roof}_m(\lambda)$.

Definition 2.6. Let λ be an e-restricted partition, $J_0(\lambda)$ the corresponding set of beta numbers of charge 0. We write J_0 for $J_0(\lambda)$ and define

$$U(J_0) = \{x \in J_0 \mid x - e \notin J_0\}.$$

If λ is an e-core then we define $\mathrm{down}_0(\lambda) = \lambda$. Otherwise let $p' = \min U(J_0)$ and define

$$W(J_0) = \{x > p' - e \mid x \in J_0, x + e \notin J_0\} \cup \{p'\}.$$

It is clear that $W(J_0)$ is nonempty. Let $q' = \min W(J_0)$. Then we define

$$\mathrm{down}(J_0) = (J_0 \setminus \{q'\}) \sqcup \{p' - e\}$$

and we denote the corresponding partition by $\mathrm{down}_0(\lambda)$.

In [4], it is shown that $\mathrm{down}_0(\lambda)$ is again e-restricted and we reach an e-core after applying down_0 finitely many times.

Definition 2.7. Let λ be an e-restricted partition. Apply down_0 repeatedly until we reach an e-core. We denote the resulting e-core by $\mathrm{base}_0(\lambda)$.

Finally, let λ be an e-restricted partition, $J_0^{\max}$ the set of beta numbers of charge 0 for $\mathrm{base}_0(\lambda)$. Define $M_i(\lambda)$, for $i \in \mathbb{Z}/e\mathbb{Z}$, by

$$M_i(\lambda) = \max\{x \in J_0^{\max} \mid x + e\mathbb{Z} = i\}.$$

We write $M_i(\lambda)$ in decreasing order

$$M_{i_1}(\lambda) > M_{i_2}(\lambda) > \cdots > M_{i_e}(\lambda).$$

Then $J_0^{\max} \cup \{M_{i_k}(\lambda) + e\}_{1 \leqslant k \leqslant m}$ is the set of beta numbers of charge m, for some partition. We denote the partition by $\tau_m(\mathrm{base}_0(\lambda))$.

Now, the characterization of Kleshchev bipartitions is as follows.

Theorem 2.8 ([4])**.** *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition such that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. Then $\boldsymbol{\lambda}$ is Kleshchev if and only if*

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})).$$

2.3. **The Dipper-James-Murphy conjecture.** We recall the dominance order for bipartition. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ be bipartitions. In this paper, we write $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$ if

$$\sum_{k=1}^{j} \lambda_k^{(1)} \geqslant \sum_{k=1}^{j} \mu_k^{(1)} \text{ and } |\lambda^{(1)}| + \sum_{k=1}^{j} \lambda_k^{(2)} \geqslant |\mu^{(1)}| + \sum_{k=1}^{j} \mu_k^{(2)},$$

for all $j \geqslant 0$.

Definition 2.9. Let $\boldsymbol{\lambda}$ be a bipartition of rank n. A *standard bitableau of shape* $\boldsymbol{\lambda}$ is a sequence of bipartitions

$$\boldsymbol{\emptyset} = \boldsymbol{\lambda}[0] \subseteq \boldsymbol{\lambda}[1] \subseteq \cdots \subseteq \boldsymbol{\lambda}[n] = \boldsymbol{\lambda}$$

such that the rank of $\boldsymbol{\lambda}[k]$ is k, for $0 \leqslant k \leqslant n$. Let $\mathbf{t}$ be a standard bitableau of shape $\boldsymbol{\lambda}$. Then the *residue sequence* of $\mathbf{t}$ is the sequence

$$(\mathrm{res}(\gamma[1]), \ldots, \mathrm{res}(\gamma[n])) \in (\mathbb{Z}/e\mathbb{Z})^n$$

where $\gamma[k] = \boldsymbol{\lambda}[k]/\boldsymbol{\lambda}[k-1]$, for $1 \leqslant k \leqslant n$.

A standard bitableau may be viewed as filling of the nodes of $\boldsymbol{\lambda}$ with numbers $1, \ldots, n$: we write k in the node $\gamma[k]$, for $1 \leqslant k \leqslant n$.

Definition 2.10. A bipartition $\boldsymbol{\lambda}$ is $(-q^m, e)$*-restricted*, or *restricted* for short, if there exists a standard bitableau $\mathbf{t}$ of shape $\boldsymbol{\lambda}$ such that the residue sequence of any standard bitableau of shape $\boldsymbol{\nu} \triangleleft \boldsymbol{\lambda}$ does not coincide with the residue sequence of $\mathbf{t}$.

Conjecture 2.11 ([7, Conj. 8.13])**.** *A bipartition $\boldsymbol{\lambda}$ is Kleshchev if and only if it is restricted.*

3. Properties of Kleshchev bipartitions

The aim of this section is to prove some combinatorial results concerning Kleshchev bipartitions.

3.1. **Admissible sequence.**

Definition 3.1. Let $i \in \mathbb{Z}/e\mathbb{Z}$. We say that a sequence of removable i-nodes $R_1, ..., R_s$ (where $s \geq 1$) of a bipartition $\boldsymbol{\lambda}$ is an *admissible sequence of i-nodes for $\boldsymbol{\lambda}$* if

- $R_1, ..., R_s$ are the lowest s removable i-nodes of $\boldsymbol{\lambda}$ and every addable i-nodes is above all of these nodes, and
- if there is a removable i-node R above $R_1, ..., R_s$, there must exist an addable i-node below R.

The following lemma shows the existence of an admissible sequence of i-nodes, for some i, for a Kleshchev bipartition: choose i as in the lemma and read addable and removable i-nodes in the total order of nodes. Suppose that $\boldsymbol{\lambda}$ has at least one addable i-node and let η be the lowest addable i-node. Then removable i-nodes below η form an admissible sequence of i-nodes. If $\boldsymbol{\lambda}$ does not have an addable i-node, all removable i-nodes of $\boldsymbol{\lambda}$ form an admissible sequence of i-nodes.

Lemma 3.2. *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a nonempty Kleshchev bipartition. Then there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a removable i-node γ such that if η is an addable i-node of $\boldsymbol{\lambda}$ then η is above γ.*

Proof. Recall that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e-restricted. There are two cases to consider.

- Assume that $\lambda^{(2)}$ is not the empty partition. Let $\lambda^{(2)}_j$ be the last row. Define $\gamma = (j, \lambda^{(2)}_j - 1, 2)$ and $i = \mathrm{res}(\gamma)$. Since $\lambda^{(2)}$ is e-restricted, the residue of the addable node $(j + 1, 0, 2)$ is not i. Hence all the addable i-node of $\boldsymbol{\lambda}$ are above γ.
- Assume that $\lambda^{(2)}$ is the empty partition. Let $\lambda^{(1)}_j$ be the last row. Define $\gamma = (j, \lambda^{(1)}_j - 1, 1)$ and $i = \mathrm{res}(\gamma)$. Since $\lambda^{(1)}$ is e-restricted, the residue of the addable node $(j + 1, 0, 1)$ is not i. We show that the residue of the addable node $(0, 0, 2)$ is not i. Suppose to the contrary that the residue is i. As $\boldsymbol{\lambda}$ is Kleshchev, we may delete good nodes successively to obtain the empty bipartition. Hence γ must be deleted at some point in the process. However, it can never be a good node since we always have an addable i-node $(0, 0, 2)$ just below it and there is no removable i-node between them, so we have a contradiction.

Hence the claim follows. □

Lemma 3.3. *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a nonempty Kleshchev bipartition and $R_1, ..., R_s$ an admissible sequence of i-nodes for $\boldsymbol{\lambda}$. Define $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$ by $\boldsymbol{\lambda} = \boldsymbol{\mu} \sqcup \{R_1, ..., R_s\}$. Then $\boldsymbol{\mu}$ is Kleshchev.*

Proof. Recall that $\lambda^{(1)}$ and $\lambda^{(2)}$ are both e-restricted. We claim that $\mu^{(1)}$ and $\mu^{(2)}$ are both e-restricted. We only prove that $\mu^{(1)}$ is e-restricted as the proof for $\mu^{(2)}$ is the same. Suppose that $\mu^{(1)}$ is not e-restricted. Since $\lambda^{(1)}$ is e-restricted, it occurs only when there exists j such that $\lambda^{(1)}_j = \lambda^{(1)}_{j+1} + e - 1$, $\mu^{(1)}_j = \lambda^{(1)}_j$, $\mu^{(1)}_{j+1} = \lambda^{(1)}_{j+1} - 1$ and $\mathrm{res}(j+1, \lambda^{(1)}_{j+1} - 1, 1) = i$. But then $\mathrm{res}(j, \lambda^{(1)}_j - 1, 1) = i$, which implies $\mu^{(1)}_j = \lambda^{(1)}_j - 1$ by definition of $\boldsymbol{\mu}$.

First case. First we consider the case when either $\lambda^{(2)} = \emptyset$ or $\lambda^{(2)} \neq \emptyset$ and $\lambda^{(2)}$ has no addable i-node. If $\lambda^{(1)}$ has no addable i-node then the admissible sequence $R_1, ..., R_s$ is given by all the removable i-nodes of $\boldsymbol{\lambda}$, and thus all the normal i-nodes of $\boldsymbol{\lambda}$. Hence $\boldsymbol{\mu} = \tilde{e}_i^s \boldsymbol{\lambda}$, which implies that $\boldsymbol{\mu}$ is Kleshchev. Therefore, we may and do assume that $\lambda^{(1)}$ has at least one addable i-node.

Define $t \geq 0$ by

$$t = \min\{k \mid \mathrm{roof}_m(\lambda^{(1)}) = \mathrm{up}_m^k(\lambda^{(1)})\}.$$

We prove that $\boldsymbol{\mu}$ is Kleshchev by induction on t. Note that if $\boldsymbol{\lambda}$ is Kleshchev then so is $(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)})$ since

$$\mathrm{roof}_m(\mathrm{up}_m(\lambda^{(1)})) = \mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})).$$

Suppose that $t = 0$. Then $\lambda^{(1)}$ is an e-core. As $\lambda^{(1)}$ has an addable i-node, $\lambda^{(1)}$ has no removable i-node. Thus all the removable i-nodes of $\boldsymbol{\lambda}$ are nodes of $\lambda^{(2)}$. As $\lambda^{(2)}$ has no addable i-node, the admissible sequence $R_1, ..., R_s$ is given by all the normal i-nodes of $\boldsymbol{\lambda}$. Hence, $\boldsymbol{\mu} = \tilde{e}_i^s \boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is Kleshchev.

Suppose that $t > 0$ and that the lemma holds for $(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)})$. Recall that we have assumed that $\lambda^{(1)}$ has an addable i-node. Let r be the minimal addable i-node of $J_m := J_m(\lambda^{(1)})$. The corresponding addable i-node of $\lambda^{(1)}$, say γ, is the lowest addable i-node of $\boldsymbol{\lambda}$ and the admissible sequence $R_1, ..., R_s$ is given by all the removable i-nodes of $\boldsymbol{\lambda}$ that is below γ. If there is no removable i-node greater than r then all the removable i-nodes of $\boldsymbol{\lambda}$ are normal and by deleting them, we obtain that $\boldsymbol{\mu}$ is Kleshchev. Hence, we assume that there is a removable i-node greater than r. As $r + 1 \notin J_m$, this implies that there is $x \in U(J_m)$ on the $(i+1)^{th}$ runner such that $x > r + 1$. Let $p = \max U(J_m)$. Then $x \in U(J_m)$ implies that $p \geq x > r + 1$. As p moves to $q > p$, it implies that $R_1, ..., R_s$ is an admissible sequence of i-nodes for $(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)})$ and that

$$(\mathrm{up}_m(\lambda^{(1)}), \lambda^{(2)}) = (\mathrm{up}_m(\mu^{(1)}), \mu^{(2)}) \sqcup \{R_1, ..., R_s\}.$$

Now, $(\mathrm{up}_m(\mu^{(1)}), \mu^{(2)})$ is Kleshchev by the induction hypothesis. Hence,

$$\mathrm{roof}_m(\mu^{(1)}) = \mathrm{roof}_m(\mathrm{up}_m(\mu^{(1)})) \subseteq \tau_m(\mathrm{base}_0(\mu^{(2)}))$$

and $\boldsymbol{\mu}$ is Kleshchev as desired.

Second case : Now, we consider the case when $\lambda^{(2)} \neq \emptyset$ and $\lambda^{(2)}$ has at least one addable i-node. Note that it forces $\lambda^{(1)} = \mu^{(1)}$ and $R_1, ..., R_s$ are nodes of $\lambda^{(2)}$. Define $t' \geq 0$ by

$$t' = \min\{k \mid \mathrm{base}_0(\lambda^{(2)}) = \mathrm{down}_0^k(\lambda^{(2)})\}.$$

We prove that $\boldsymbol{\mu}$ is Kleshchev by induction on t'. Note that if $\boldsymbol{\lambda}$ is Kleshchev then so is $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$ since

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})) = \tau_m(\mathrm{base}_0(\mathrm{down}_0(\lambda^{(2)}))).$$

If $t' = 0$ then $\lambda^{(2)}$ is an e-core and it has removable i-nodes $R_1, ..., R_s$. Hence, $\lambda^{(2)}$ has no addable i-node and we are reduced to the previous case. Thus we suppose that $t' > 0$ and that the lemma holds for $(\lambda^{(1)}, \mathrm{down}(\lambda^{(2)}))$. Let $J = J_0(\lambda^{(2)})$ and

$$r = \min\{x \in J \mid x + e\mathbb{Z} = i + 1,\ x - 1 \notin J\} - 1.$$

Note that r is on the i^{th} runner. Then there exists $N \geq 1$ such that

$$r, r+e, \ldots, r+(N-1)e \notin J \quad \text{and} \quad r + Ne \in J.$$

Let $p' = \min U(J)$. Then $p' \leq r + Ne$. Suppose that p' is not on the i^{th} runner or the $(i+1)^{th}$ runner. If a node which is not on one of these two runners moves to $p' - e$ by the down operation, the admissible sequence $R_1, ..., R_s$ is an admissible sequence of i-nodes for $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$ and

$$(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)})) = (\mu^{(1)}, \mathrm{down}_0(\mu^{(2)})) \sqcup \{R_1, ..., R_s\}.$$

Thus, by the induction hypothesis, $(\mu^{(1)}, \mathrm{down}_0(\mu^{(2)}))$ is Kleshchev. Hence,

$$\mathrm{roof}_m(\mu^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\mathrm{down}_0(\mu^{(2)}))) = \tau_m(\mathrm{base}_0(\mu^{(2)}))$$

implies that $\boldsymbol{\mu}$ is Kleshchev.

If a node in one of the two runners moves to $p' - e$, then there exists $0 \leq k \leq N - 1$ such that $r + ke + 1 \in J$, $r + (k+1)e + 1 \notin J$ and $r + ke + 1$ moves to $p' - e$. Suppose that $k < N - 1$. Then, $r + ke \in J_0(\mu^{(2)})$ and $r + (k+1)e \notin J_0(\mu^{(2)})$. Hence, $r + ke$ moves to $p' - e$ to obtain $\mathrm{down}_0(\mu^{(2)})$. As $r + ke + 1$ corresponds to one of $R_1, ..., R_s$, say R_k, $R_1, \ldots, \hat{R_k}, \ldots, R_s$ is an admissible sequence of i-nodes for $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$ and

$$(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)})) = (\mu^{(1)}, \mathrm{down}_0(\mu^{(2)})) \sqcup \{R_1, \ldots, \hat{R_k}, \ldots, R_s\}.$$

Hence, $(\mu^{(1)}, \mathrm{down}_0(\mu^{(2)}))$ is Kleshchev by the induction hypothesis, and $\boldsymbol{\mu}$ is Kleshchev as before. Next suppose that $k = N - 1$. As $r + (N-1)e + 1$ moves to $p' - e$, we have

$$r + (N-1)e + 1 < p' < r + Ne,$$

$r + Ne + 1 \notin J_0$ and $r + Ne$ is an addable i-node. Let K be the set of beta numbers of charge 0 of $\mu^{(2)}$. For $x \in \mathbb{Z}$, we denote $J_{\leq x} := J \cap \mathbb{Z}_{\leqslant x}$ and $K_{\leq x} := K \cap \mathbb{Z}_{\leqslant x}$. We claim that

$$\mathrm{base}(J_{\leq r+Ne}) = \mathrm{base}(K_{\leq r+Ne}).$$

Let $p' = y_0 < y_1 < \cdots < y_l < r + Ne$ be the nodes in J which are greater than or equal to p' and smaller than $r + Ne$. We show that

$$\mathrm{base}(J_{\le y_j}) = s_i\,\mathrm{base}(K_{\le y_j}) \supseteq \mathrm{base}(K_{\le y_j}),$$

for $0 \le j \le l$, where s_i means swap of the i^{th} and $(i+1)^{th}$ runners. $J_{\le p'}$ and $K_{\le p'}$ are s_i-cores in the sense of [4], and direct computation shows the formula for $j = 0$. Now we use $\mathrm{base}(J_{\le y_{j+1}}) = \mathrm{base}(\{y_{j+1}\} \cup \mathrm{base}(J_{\le y_j}))$ and $\mathrm{base}(K_{\le y_{j+1}}) = \mathrm{base}(\{y_{j+1}\} \cup \mathrm{base}(K_{\le y_j}))$ [4] and continue the similar computation and comparison of $\mathrm{base}(J_{\le y_j})$ and $\mathrm{base}(K_{\le y_j})$. At the end of the inductive step, we get

$$\mathrm{base}(J_{<r+Ne}) = s_i\,\mathrm{base}(K_{<r+Ne}) \supseteq \mathrm{base}(K_{<r+Ne}).$$

Now, one more direct computation shows

$$\mathrm{base}(\{r+Ne\} \cup \mathrm{base}(J_{<r+Ne})) = \mathrm{base}(\{r+Ne\} \cup \mathrm{base}(K_{<r+Ne})),$$

and we have the claim. Therefore, $\mathrm{base}_0(\lambda^{(2)}) = \mathrm{base}_0(\mu^{(2)})$ and we have

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})) = \tau_m(\mathrm{base}_0(\mu^{(2)})).$$

Recalling that $\lambda^{(1)} = \mu^{(1)}$, we have that $\boldsymbol{\mu}$ is Kleshchev.

Suppose that p' is on one of the two runners. As $p' \le r + Ne$, we have either $p' = r + Ne$ or $p' = r + ke + 1$, for some $0 \le k \le N - 1$. If the latter occurs, $\mathrm{down}_0(\lambda^{(2)})$ is obtained by moving a node outside the two runners to $p'-e$ or moving p' to $p'-e$, and $\mathrm{down}_0(\mu^{(2)})$ is obtained from $\mu^{(2)}$ by moving the same node outside the two runners to $p'-1-e$ or moving $p'-1$ to $p'-1-e$, respectively. Thus, $\mathrm{down}_0(\mu^{(2)})$ is obtained from $\mathrm{down}_0(\lambda^{(2)})$ by removing the nodes of an admissible sequence of i-nodes for $(\lambda^{(1)}, \mathrm{down}_0(\lambda^{(2)}))$. Hence $(\mu^{(1)}, \mathrm{down}_0(\mu^{(2)}))$ is Kleshchev by the induction hypothesis, and it follows that $\boldsymbol{\mu}$ is Kleshchev. If $p' = r + Ne$ and $r + Ne + 1 \in J_0$ then the same is true, and if $p' = r + Ne$ and $r + Ne + 1 \notin J_0$ then $\mu^{(2)} = \mathrm{down}_0^N(\lambda^{(2)})$ and we have

$$\mathrm{roof}_m(\lambda^{(1)}) \subseteq \tau_m(\mathrm{base}_0(\lambda^{(2)})) = \tau_m(\mathrm{base}_0(\mu^{(2)})).$$

Hence, $\boldsymbol{\mu} = (\lambda^{(1)}, \mu^{(2)})$ is Kleshchev. □

3.2. **Optimal sequence of a Kleshchev bipartition.** Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a Kleshchev bipartition. By the previous lemma, we may define by induction a sequence of Kleshchev bipartitions

$$\boldsymbol{\lambda} =: \boldsymbol{\lambda}[1],\ \boldsymbol{\lambda}[2], ...,\ \boldsymbol{\lambda}[p],\ \boldsymbol{\lambda}[p+1] = \boldsymbol{\emptyset}$$

and a sequence of residues

$$\underbrace{i_1, ..., i_1}_{a_1 \text{ times}}, ..., \underbrace{i_p, ..., i_p}_{a_p \text{ times}}$$

such that, for $1 \le j \le p$, $\boldsymbol{\lambda}[j] = \boldsymbol{\lambda}[j+1] \sqcup \{R_1^j, ..., R_{a_j}^j\}$ and $R_1^j, ..., R_{a_j}^j$ is an admissible sequence of i_j-nodes for $\boldsymbol{\lambda}[j]$.

[4] See [4, Prop 7.8].

We call $\underbrace{i_1, ..., i_1}_{a_1 \text{ times}}, ..., \underbrace{i_p, ..., i_p}_{a_p \text{ times}}$ an *optimal sequence* of $\boldsymbol{\lambda}$.

Example 3.4. Keeping example 2.1, it is easy to see that $((3,2),(4,2,1))$ is a Kleshchev bipartition and an optimal sequence is given by

$$2, 2, 0, 3, 3, 2, 1, 1, 0, 0, 3, 2.$$

4. Proof of the conjecture

4.1. **The result.** We are now ready to prove the conjecture. As is explained in the introduction, it is enough to prove that Kleshchev bipartitions are restricted bipartitions. To do this, we introduce certain reverse lexicographic order on the set of bipartitions.

Definition 4.1. We write $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$ if either

- there exists $j \geq 0$ such that $\lambda_k^{(2)} = \nu_k^{(2)}$, for $k > j$, and $\lambda_j^{(2)} < \nu_j^{(2)}$, or
- there exists $j \geq 0$ such that $\lambda^{(2)} = \nu^{(2)}$, $\lambda_k^{(1)} = \nu_k^{(1)}$, for $k > j$, and $\lambda_j^{(1)} < \nu_j^{(1)}$.

It is clear that if $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$ then $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$. Recall that the deformed Fock space is given a specific $U_v(\hat{sl}_e)$-module structure which is suitable for the Dipper-James-Murphy's Specht module theory.

Proposition 4.2. *Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be Kleshchev and let*

$$\underbrace{i_1, ..., i_1}_{a_1 \textit{ times}}, ..., \underbrace{i_p, ..., i_p}_{a_p \textit{ times}}$$

be an optimal sequence of $\boldsymbol{\lambda}$. Then we have

$$f_{i_1}^{(a_1)} ... f_{i_p}^{(a_p)} \boldsymbol{\emptyset} = \boldsymbol{\lambda} + \sum_{\boldsymbol{\nu} \prec \boldsymbol{\lambda}} c_{\boldsymbol{\nu}, \boldsymbol{\lambda}}(v) \boldsymbol{\nu},$$

for some Laurent polynomials $c_{\boldsymbol{\nu},\boldsymbol{\lambda}}(v) \in \mathbb{Z}_{\geq 0}[v, v^{-1}]$, in the deformed Fock space.

Proof. First note that the coefficient of $\boldsymbol{\lambda}$ is one because each admissible sequence of i_j-nodes is a sequence of normal i_j-nodes.

Now the proposition is proved by induction on p. Let $R_1^1, ..., R_{a_1}^1$ be the admissible sequence of i_1-nodes for $\boldsymbol{\lambda}$ and let $\boldsymbol{\lambda}'$ be the Kleshchev bipartition such that

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}' \sqcup \{R_1^1, ..., R_{a_1}^1\}.$$

By the induction hypothesis, we have

$$f_{i_2}^{(a_2)} ... f_{i_p}^{(a_p)} \emptyset = \boldsymbol{\lambda}' + \sum_{\boldsymbol{\nu}' \prec \boldsymbol{\lambda}'} c_{\boldsymbol{\nu}, \boldsymbol{\lambda}'}(v) \boldsymbol{\nu}'.$$

Let $\boldsymbol{\nu} \neq \boldsymbol{\lambda}$ be a bipartition such that it appears in $f_{i_1}^{(a_1)}...f_{i_p}^{(a_p)}\emptyset$ with nonzero coefficient. Then there exist removable i_1-nodes R'^1_1,...,$R'^1_{a_1}$ of $\boldsymbol{\nu}$ and a bipartition $\boldsymbol{\nu}' \preceq \boldsymbol{\lambda}'$ such that

$$\boldsymbol{\nu} = \boldsymbol{\nu}' \sqcup \{R'^1_1, ..., R'^1_{a_1}\}.$$

As $R^1_1, ..., R^1_{a_1}$ are the lowest a_1 i_1-nodes of $\boldsymbol{\lambda}$, $\boldsymbol{\nu}' = \boldsymbol{\lambda}'$ implies $\boldsymbol{\nu} \preceq \boldsymbol{\lambda}$. Hence we may assume $\boldsymbol{\nu}' \prec \boldsymbol{\lambda}'$. Suppose that we have $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$. If $\nu'^{(2)} \neq \lambda'^{(2)}$ then choose t such that $\nu'^{(2)}_t < \lambda'^{(2)}_t$ and $\nu'^{(2)}_j = \lambda'^{(2)}_j$, for $j > t$. Then we can show

(i) $\nu^{(2)}_j = \lambda^{(2)}_j$, for $j > t$,

(ii) $\nu^{(2)}_{t+1} < \nu^{(2)}_t = \nu'^{(2)}_t + 1 = \lambda'^{(2)}_t = \lambda^{(2)}_t$,

(iii) at least one of the nodes $R^1_1, ..., R^1_{a_1}$ is above $(t, \lambda^{(2)}_t - 1, 2)$.

The condition (ii) implies that $\mathrm{res}(t, \lambda^{(2)}_t - 1, 2) = \mathrm{res}(t, \nu'^{(2)}_t, 2) = i_1$. Thus $(t, \lambda^{(2)}_t - 1, 2)$ is an i_1-node of $\lambda^{(2)}$. Then (iii) implies that it is not a removable node of $\lambda^{(2)}$ (otherwise it has to be removed to obtain $\boldsymbol{\lambda}'$). This implies that $\lambda^{(2)}_t = \lambda^{(2)}_{t+1}$. Thus $\nu^{(2)}_{t+1} < \lambda^{(2)}_t = \lambda^{(2)}_{t+1}$ and (i) is contradicted.

If $\nu'^{(2)} = \lambda'^{(2)}$ then choose t such that $\nu'^{(1)}_t < \lambda'^{(1)}_t$ and $\nu'^{(1)}_j = \lambda'^{(1)}_j$, for $j > t$. Then we argue as above. □

Corollary 4.3. *The Dipper-James-Murphy conjecture is true.*

Proof. Observe that $\boldsymbol{\nu}$ appears in $f_{i_n} \cdots f_{i_1}\emptyset$ if and only if there exists a standard bitableau of shape $\boldsymbol{\nu}$ such that its residue sequence is $(i_1, \ldots, i_n)$. Let $\boldsymbol{\lambda}$ be Kleshchev. Then Proposition 4.2 shows that there is a standard bitableau $\mathbf{t}$ such that if the residue sequence of $\mathbf{t}$ appears as the residue sequence of a standard bitableau of shape $\boldsymbol{\nu}$ then $\boldsymbol{\nu} \preceq \boldsymbol{\lambda}$. Suppose that the residue sequence of $\mathbf{t}$ is the residue sequence of a standard bitableau of shape $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$. As $\boldsymbol{\nu} \lhd \boldsymbol{\lambda}$ implies $\boldsymbol{\lambda} \prec \boldsymbol{\nu}$, we cannot have $\boldsymbol{\nu} \preceq \boldsymbol{\lambda}$, a contradiction. Hence $\boldsymbol{\lambda}$ is restricted. □

4.2. **Remark.** We conclude the paper with a remark.

Remark 4.4. There is a systematic way to produce realizations of the crystal $B(\Lambda_0 + \Lambda_m)$ on a set of bipartitions, for each choice of $\log_q(-Q)$. The bipartitions are called Uglov bipartitions. Recent results of Geck [8] and Geck and the second author [9] show that Uglov bipartitions naturally label simple $\mathcal{H}_n$-modules, and Rouquier's theory of the BGG category of rational Cherednik algebras as quasihereditary covers of Hecke algebras naturally explains the existence of various Specht module theories which depends on $\log_q(-Q)$.

We conjecture that Uglov bipartitions satisfy an analogue of Proposition 4.2 except that the dominance order is replaced by an appropriate a-value in the sense of [9, Prop 2.1]. As is mentioned in the introduction, it is known that this conjecture is true in the case where Uglov bipartitions are FLOTW bipartitions [10, Prop. 4.6].

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